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AN EFFICIENT ALGORITHM FOR COMPUTING THE G GUIDANCE MATRIX

by

W. L. DAVIS
Strategic Systems Department

JANUARY 1979

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This paper derives a new method for computing the Q-guidance matrix. This matrix is defined as the derivative with respect to position of the correlated velocity $\vec{V}_C$; i.e., the velocity required to reach a given target in a specified time in an inverse-square gravitational field. The method is completely analytic and yields a simple and efficient algorithm.

The analytic equation for the partial derivative of the horizontal component of correlated velocity with respect to range angle developed here is also

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of independent interest. To the author's knowledge, no such expression for this quantity in terms of the given conditions and the components of $\vec{V}_C$ has been derived previously. This equation, in conjunction with the others given here, makes it possible to compute virtually any (first order) derivative of $\vec{V}_C$ of interest.

The Q-matrix algorithm is of use primarily in the analysis and modeling of guidance systems using some form of $\vec{V}_g$ steering.

FOREWORD

This report describes an analytic method of computing the guidance Q-matrix which is much simpler and more efficient than methods used previously. The work was performed in the Fire Control Presetting Analysis Branch (K51), FBM Geoballistics Division, Strategic Systems Department, and was authorized under Strategic Systems Project Office Task Assignment 36401.

The author would like to thank Mr. T. Alexander for permission to include his derivation of the algorithm for computing correlated velocity given in Appendix A. He is also indebted to Mr. J. F. Ray for reviewing much of the original work and correcting an error in one of the key equations.

This report has been reviewed and approved by D. L. Owen; J. R. Fallin, Head, Fire Control Presetting Analysis Branch; and C. W. Duke, Head, FBM Geoballistics Division.

Released by:

R. A. NIEMANN, Head
Strategic Systems Department

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INTRODUCTION

PROBLEM DESCRIPTION

Consider the motion of a body (or point mass) in an inverse-square gravitational field centered at the origin of some arbitrary inertial reference frame. At any point $\vec{R}$ the body will experience an acceleration of $-K^2R^{-3}\vec{R}$, where K^2 is the gravitational constant for the given field.

Now suppose two nonparallel position vectors $\vec{R}$, $\vec{R}_T$ and a time t_f are given. The <u>correlated velocity</u> (denoted $\vec{V}_C$) for these conditions is defined as the velocity at $\vec{R}$ which will cause a body to reach $\vec{R}_T$ in time t_f if no other force acts on it. Clearly $\vec{V}_C$ is determined by $\vec{R}$, $\vec{R}_T$, t_f, and K^2, so we may write $\vec{V}_C = \vec{V}_C(t_f, \vec{R}, \vec{R}_T, K^2)$. The symbols $\partial\vec{V}_C/\partial\vec{R}$, $\partial\vec{V}_C/\partial t_f$, etc., will be used to denote the partials obtained by holding the other parameters in this set constant. In particular, $\partial\vec{V}_C/\partial\vec{R}$ is commonly called the Q-matrix, and denoted simply by Q.

It is well known that the path of a body in any inverse-square force field will lie along a conic section with the center of force as a focus. We will consider only the case in which the path is along an ellipse and subtends an angle less than π. $\vec{V}_C$ will be considered undefined if no value corresponding to such a path exists. This restriction assures that $\vec{V}_C$ is well-defined when it exists.

This paper describes a new method for computing the Q-matrix which is completely analytic, yet is also simple and efficient.

BASIC CONCEPTS AND TERMINOLOGY

Clearly $\vec{R}$ and $\vec{R}_T$ determine the plane of the trajectory. It is convenient to define an orthogonal coordinate system (which will be called the <u>local coordinate frame</u>) with the axes

$\vec{U}_1$ = unit $(\vec{R})$,

$\vec{U}_2$ = unit $[(\vec{R} \times \vec{R}_T) \times \vec{R}]$, and

$\vec{U}_3 = \vec{U}_1 \times \vec{U}_2$.

We then define $V_{CR} = \vec{U}_1 \cdot \vec{V}_C$ and $V_{C\theta} = \vec{U}_2 \cdot \vec{V}_C$. $(\vec{U}_3 \cdot \vec{V}_C = 0)$. Similarly, $V_R = \vec{U}_1 \cdot \vec{V}$ and $V_\theta = \vec{U}_2 \cdot \vec{V}$, where $\vec{V}$ is the actual velocity at $\vec{R}$. The range angle, θ_R, is the angle between $\vec{R}$ and $\vec{R}_T$. Finally, the difference $\vec{V}_C - \vec{V}$ will be called the velocity to be gained and denoted by $\vec{V}_g$.

Since V_{CR} and $V_{C\theta}$ are scalars, it is often desirable to express them as functions of scalars. One such set is $(t_f, R, R_T, \theta_R, K^2)$; unless otherwise indicated V_{CR} and $V_{C\theta}$ will be treated as functions of these parameters. Thus $\partial V_{C\theta}/\partial t_f$, $\partial V_{CR}/\partial \theta_R$, etc., will denote the partials obtained by holding the other parameters in this set constant.

The usual point of view is that $\vec{R}$ is the current position and $\vec{R}_T$ the position at which the "target" will be located after an elapsed time t_f (often called the "time-to-go"). Thus $\vec{V}_C$ is the velocity required to reach the target in the specified time. With this interpretation it makes sense to consider $\vec{R}$, $\vec{V}_C$, V_{CR}, and $V_{C\theta}$ as functions of time (t). In this case it is understood that $t = 0$ at $\vec{R}$ and t_f decreases with time; i.e., $dt_f/dt = -1$. (The idea is that we want to reach the target at a specified time.) Of course $V_{C\theta}$ and V_{CR} are always defined relative to the current position; that is, $V_{CR}(t) = V_C(t) \cdot \text{unit}[R(t)]$, and similarly for $V_{C\theta}$. Unless otherwise indicated, it is assumed that $\vec{V} = \vec{V}_C$. Note that, if $\vec{V}(0) = \vec{V}_C(0)$ and no force but gravity is acting on the body, then $\vec{V}(t) = \vec{V}_C(t)$ for $0 \leq t \leq t_f$.

APPLICATIONS

The algorithm developed in this paper is useful in the analysis of guidance systems using $\vec{V}_g$ steering. In particular, it can be used to compute the time derivative of $\vec{V}_g$ by means of equation B1, which is derived in Appendix B.

The equations for $\partial V_{C\theta}/\partial t_f$ and $\partial V_{C\theta}/\partial \theta_R$ are also of interest in their own right.

The algorithm for computing Q and the equation for $\partial V_{C\theta}/\partial \theta_R$ are used in several trajectory simulation programs developed by NSWC.

DEVELOPMENT OF Q-MATRIX ALGORITM

PRELIMINARY RESULTS

$\vec{V}_g$ steering and the Q-matrix have both been studied extensively, and some earlier results will be useful here.

1. An efficient algorithm for computing t_f for a given value of $V_{C\theta}$ was developed by T. Alexander. Differentiation of the equations involved also yields an analytic expression for $\partial V_{C\theta}/\partial t_f$. A derivation of these equations, together with a suggested iteration method for implementing them, is given in Appendix A.

One of the equations developed in this derivation, e.g. (A5), can be rearranged to give a useful formula for V_{CR}:

$$V_{CR} = \left[\left(\cos\theta_R - \frac{R}{R_T}\right)V_{C\theta} + \frac{K^2}{RV_{C\theta}}\left(1 - \cos\theta_R\right)\right]/\sin\theta_R \qquad (1)$$

2. The Q-matrix is symmetric.*

DERIVATION

The algorithm will be developed in two stages. First, a representation for Q in the local coordinate frame in terms of $V_{C\theta}$, V_{CR}, and certain of their derivatives will be obtained. Second, formulas for the required derivatives will be derived.

To obtain the representation of Q, fix $\vec{R}_T$ and t_f, and a position $\vec{R}_0$ at which Q is to be evaluated. Let $\vec{U}_1^0$, $\vec{U}_2^0$, and $\vec{U}_3^0$ be the axes of the local

* C. J. Cohen and R. H. Lyddane, The Q-Guidance Matrix and Its Symmetry (U), Naval Proving Ground, Dahlgren, Va., TR K-6/59, June 1959. UNCLASSIFIED

coordinate system $\mathcal{L}$ defined at $\vec{R}_0$, and let θ_{R0} be the corresponding range angle. For an arbitrary position $\vec{R}$, let

$$R_R = \vec{R} \cdot \vec{U}_1^0,$$

$$R_\theta = \vec{R} \cdot \vec{U}_2^0, \text{ and}$$

$$R_P = \vec{R} \cdot \vec{U}_3^0 .$$

For any vector $\vec{X}$, the representation of $\vec{X}$ in $\mathcal{L}$ will be denoted by $\vec{X}^{\mathcal{L}}$, and the corresponding representation for Q by $Q^{\mathcal{L}}$.

N.B.: While the axes $\vec{U}_1$, $\vec{U}_2$, and $\vec{U}_3$ are always defined relative to $\vec{R}$, the symbol $\mathcal{L}$ denotes the fixed frame $\vec{U}_1^0$, $\vec{U}_2^0$, $\vec{U}_3^0$.

By definition

$$Q^{\mathcal{L}} = \left(\frac{\partial \vec{V}_C^{\mathcal{L}}}{\partial R_R}, \frac{\partial \vec{V}_C^{\mathcal{L}}}{\partial R_\theta}, \frac{\partial \vec{V}_C^{\mathcal{L}}}{\partial R_P} \right)_{\vec{R} = \vec{R}_0} . \tag{2}$$

Also, it follows immediately from the definitions that $\vec{V}_C = V_{CR} \vec{U}_1 + V_{C\theta} \vec{U}_2$. Applying the chain rule to this equation yields

$$\frac{\partial \vec{V}_C}{\partial R_i} = \frac{\partial V_{CR}}{\partial R_i} \vec{U}_1 + V_{CR} \frac{\partial \vec{U}_1}{\partial R_i} + \frac{\partial V_{C\theta}}{\partial R_i} \vec{U}_2 + V_{C\theta} \frac{\partial \vec{U}_2}{\partial R_i} , \tag{3}$$

where R_i is any component of $\vec{R}$ (e.g., R_R, R_θ, R_P).

It is easily verified that

$$\left. \begin{aligned} \vec{U}_1 &= R^{-1} \vec{R} \\ \vec{U}_2 &= (\hat{R}_T - R^{-1} \cos \theta_R \vec{R})/\sin \theta_R, \text{ where } \hat{R}_T = \text{unit}(\vec{R}_T) . \end{aligned} \right\} \tag{4}$$

Since $\hat{R}_T^{\mathcal{L}} = (\cos \theta_{R0}, \sin \theta_{R0}, 0)^T$, in $\mathcal{L}$ these equations take the form

$$\vec{U}_1^{\mathcal{L}} = \begin{pmatrix} R^{-1} R_R \\ R^{-1} R_\theta \\ R^{-1} R_P \end{pmatrix}; \vec{U}_2^{\mathcal{L}} = \begin{bmatrix} (\cos \theta_{R0} - R^{-1} R_R \cos \theta_R) \csc \theta_R \\ (\sin \theta_{R0} - R^{-1} R_\theta \cos \theta_R) \csc \theta_R \\ -R^{-1} R_P \cot \theta_R \end{bmatrix} . \tag{5}$$

Now $R^2 = R_R^2 + R_\theta^2 + R_P^2$ and $\cos\theta_R = \vec{U}_1 \cdot \hat{R}_T = (R_R \cos\theta_{R0} + R_\theta \sin\theta_{R0})R^{-1}$. Differentiating these expressions with respect to R_R, R_θ, and R_P and evaluating the results at $\vec{R} = \vec{R}_0$ yields

$$\frac{\partial R}{\partial R_\theta} = \frac{\partial R}{\partial R_P} = \frac{\partial \theta_R}{\partial R_R} = \frac{\partial \theta_R}{\partial R_P} = 0; \ \frac{\partial R}{\partial R_R} = 1; \ \frac{\partial \theta_R}{\partial R_\theta} = -R^{-1} \ . \tag{6}$$

The same operation can now be performed on Equation 5 to give (at $\vec{R}_0$):

$$\frac{\partial \vec{U}_1^{\mathcal{L}}}{\partial R_R} = 0; \ \frac{\partial \vec{U}_1^{\mathcal{L}}}{\partial R_\theta} = \begin{pmatrix} 0 \\ R^{-1} \\ 0 \end{pmatrix}; \ \frac{\partial \vec{U}_1^{\mathcal{L}}}{\partial R_P} = \begin{pmatrix} 0 \\ 0 \\ R^{-1} \end{pmatrix} \tag{7}$$

$$\frac{\partial \vec{U}_2^{\mathcal{L}}}{\partial R_R} = 0; \ \frac{\partial \vec{U}_2^{\mathcal{L}}}{\partial R_\theta} = \begin{pmatrix} -R^{-1} \\ 0 \\ 0 \end{pmatrix}; \ \frac{\partial \vec{U}_2^{\mathcal{L}}}{\partial R_P} = \begin{pmatrix} 0 \\ 0 \\ -R^{-1} \cot\theta_R \end{pmatrix} \ . \tag{8}$$

Next recall that $V_{C\theta}$ and V_{CR} are functions of t_f, R, R_T, θ_R, and K^2. Since t_f, R_T, and K^2 are constant here, Equation 6 yields immediately

$$\frac{\partial V_{Ci}}{\partial R_R} = \frac{\partial V_{Ci}}{\partial R} \ ; \ \frac{\partial V_{Ci}}{\partial R_\theta} = -R^{-1} \frac{\partial V_{Ci}}{\partial \theta_R} \ ; \ \frac{\partial V_{Ci}}{\partial R_P} = 0 \ , \tag{9}$$

where V_{Ci} can be interpreted as either $V_{C\theta}$ or V_{CR}.

Finally, substituting Equations 7, 8, and 9 into Equation 3 with R_i interpreted as R_R, R_θ, and R_P and substituting the results into Equation 2 yields

$$Q^{\mathcal{L}} = \begin{bmatrix} \dfrac{\partial V_{CR}}{\partial R} & \dfrac{-1}{R}\left(\dfrac{\partial V_{CR}}{\partial \theta_R} + V_{C\partial}\right) & 0 \\ \dfrac{\partial V_{C\theta}}{\partial R} & \dfrac{-1}{R}\left(\dfrac{\partial V_{C\theta}}{\partial \theta_R} - V_{CR}\right) & 0 \\ 0 & 0 & R^{-1}(V_{CR} - V_{C\theta} \cot\theta_R) \end{bmatrix} \ . \tag{10}$$

This is the representation of Q mentioned earlier. We now turn to the problem of finding formulas for the partials of V_{CR} and $V_{C\theta}$ with respect to R and θ_R.

First, Equation 1 gives V_{CR} as a function of $V_{C\theta}$, R, R_T, θ_R, and K^2, say $V_{CR} = f(V_{C\theta}, R, R_T, \theta_R, K^2)$. The derivatives of V_{CR} which occur in Equation 11, however, are defined with respect to the set of parameters t_f, R, R_T, θ_R, K^2, as explained in the Introduction. The partials of f can be obtained simply by differentiating Equation 1. This gives

$$\frac{\partial f}{\partial \theta_R} = \frac{K^2}{RV_{C\theta}} - V_{C\theta} - V_{CR} \cot \theta_R \tag{11}$$

$$\frac{\partial f}{\partial R} = \frac{1}{R}\left[V_{C\theta}\left(\cos \theta_R - 2 \frac{R}{R_T}\right)\csc \theta_R - V_{CR}\right] \tag{12}$$

$$\frac{\partial f}{\partial V_{C\theta}} = 2 \csc \theta_R\left(\cos \theta_R - \frac{R}{R_T}\right) - \frac{V_{CR}}{V_{C\theta}} \quad . \tag{13}$$

The chain rule gives the partials we need in terms of these:

$$\frac{\partial V_{CR}}{\partial \theta_R} = \frac{\partial f}{\partial \theta_R} + \frac{\partial f}{\partial V_{C\theta}} \frac{\partial V_{C\theta}}{\partial \theta_R} \tag{14}$$

$$\frac{\partial V_{CR}}{\partial R} = \frac{\partial f}{\partial R} + \frac{\partial f}{\partial V_{C\theta}} \frac{\partial V_{C\theta}}{\partial R} \quad . \tag{15}$$

Thus, given $\partial V_{C\theta}/\partial R$ and $\partial V_{C\theta}/\partial \theta_R$, the required partials of V_{CR} can be determined.

(It may be of interest here to note that $\partial V_{CR}/\partial t_f$, though not required to compute Q, can also be computed easily from

$$\frac{\partial V_{CR}}{\partial t_f} = \frac{\partial f}{\partial V_{C\theta}} \frac{\partial V_{C\theta}}{\partial t_f} \quad . \tag{16}$$

Thus, e.g., $\frac{\partial \vec{V}_C}{\partial t_f} = \frac{\partial V_{CR}}{\partial t_f} \vec{U}_1 + \frac{\partial V_{C\theta}}{\partial t_f} \vec{U}_2$ can be found via these equations.)

We now turn to the problem of finding $\partial V_{C\theta}/\partial R$ and $\partial V_{C\theta}/\partial \theta_R$. Recall that $\partial V_{C\theta}/\partial t_f$ can be computed directly once $V_{C\theta}$ is determined. Moreover, if two independent equations relating $\partial V_{C\theta}/\partial R$, $\partial V_{C\theta}/\partial \theta_R$, and $\partial V_{C\theta}/\partial t_f$ can be found, it may be possible to solve for $\partial V_{C\theta}/\partial R$ and $\partial V_{C\theta}/\partial \theta_R$ in terms of $\partial V_{C\theta}/\partial t_f$. One such equation can be obtained as follows.

Recall that $V_R \equiv \vec{V} \cdot \vec{U}_1$ and $V_\theta \equiv \vec{V} \cdot \vec{U}_2$, while t is the current time (<u>not</u> time-to-go). Assume that $\vec{V}$ is in the trajectory plane (so that V_θ is the horizontal component of velocity) and the only force acting is that of gravity (but not that $\vec{V} = \vec{V}_C$). Then

$$\frac{\partial V_{C\theta}}{\partial t} = \frac{\partial V_{C\theta}}{\partial R}\frac{\partial R}{\partial t} + \frac{\partial V_{C\theta}}{\partial \theta_R}\frac{\partial \theta_R}{\partial t} + \frac{\partial V_{C\theta}}{\partial t_f}\frac{\partial t_f}{\partial t} \quad . \tag{17}$$

But clearly

$$\frac{\partial R}{\partial t} = V_R; \quad \frac{\partial \theta_R}{\partial t} = -\frac{V_\theta}{R}; \quad \frac{\partial t_f}{\partial t} = -1 \quad . \tag{18}$$

Thus

$$\frac{\partial V_{C\theta}}{\partial t} = \frac{\partial V_{C\theta}}{\partial R} V_R - \frac{\partial V_{C\theta}}{\partial \theta_R}\frac{V_\theta}{R} - \frac{\partial V_{C\theta}}{\partial t_f} \quad . \tag{19}$$

Next, from conservation of the angular momentum ℓ, we have

$$V_\theta R = \ell \text{ is constant.} \tag{20}$$

Differentiating gives

$$\frac{\partial V_\theta}{\partial t} R + \frac{\partial R}{\partial t} V_\theta = 0 \quad . \tag{21}$$

Now if we set $\vec{V}_C = \vec{V}$, obviously $V_{C\theta} = V_\theta$ and $V_{CR} = V_R$. Also, since $\vec{R}$ will now follow a path reaching the target at the desired time, $\vec{V}$ will continue to equal $\vec{V}_C$; thus $\partial V_\theta/\partial t = \partial V_{C\theta}/\partial t$. Eliminating $\partial V_{C\theta}/\partial t$ between Equations 19 and 21 then yields

$$V_{CR}\frac{\partial V_{C\theta}}{\partial R} = \frac{V_{C\theta}}{R}\left(\frac{\partial V_{C\theta}}{\partial \theta_R} - V_{CR}\right) + \frac{\partial V_{C\theta}}{\partial t_f} \quad . \tag{22}$$

Since V_{CR} can be zero (i.e., at apogee), it is not legitimate to divide through by V_{CR} to obtain a formula for $\partial V_{C\theta}/\partial R$. This will cause no problem.

Equation 22 is the first of the two independent equations sought relating the partials of $V_{C\theta}$. Since the Q-matrix is symmetric, it would seem that the second relation needed can easily be obtained by setting $Q_{1,2} = Q_{2,1}$; i.e.

$$\frac{\partial V_{C\theta}}{\partial R} = -\frac{1}{R}\left(\frac{\partial V_{CR}}{\partial \theta_R} + V_{C\theta}\right) . \tag{23}$$

Combining this with Equation 14 yields a simple equation relating $\partial V_{C\theta}/\partial R$ and $\partial V_{C\theta}/\partial \theta_R$:

$$\frac{\partial V_{C\theta}}{\partial R} = -\frac{1}{R}\left(\frac{\partial f}{\partial \theta_R} + \frac{\partial f}{\partial V_{C\theta}}\frac{\partial V_{C\theta}}{\partial \theta_R} + V_{C\theta}\right) . \tag{24}$$

Unfortunately, when Equations 22 and 24 are considered as a system of linear equations in $\partial V_{C\theta}/\partial R$ and $\partial V_{C\theta}/\partial \theta_R$, the determinant of the system is zero for the minimum-energy case, which is certainly a case of interest. (See Appendix C.) So Equation 24 cannot <u>always</u> be used to determine $\partial V_{C\theta}/\partial \theta_R$ and $\partial V_{C\theta}/\partial R$. (However, note that Equation 24, unlike Equation 22, can always be used to find $\partial V_{C\theta}/\partial R$.) Thus another such relation must be found.

This new relation will be obtained indirectly by finding two independent equations relating $\partial V_{C\theta}/\partial R_T$, $\partial V_{C\theta}/\partial R$, $\partial V_{C\theta}/\partial \theta_R$, and $\partial V_{C\theta}/\partial t_f$, and eliminating $\partial V_{C\theta}/\partial R_T$ between them.

The first of these can be obtained by considering the "reverse" trajectory from $\vec{R}_T$ to $\vec{R}$ in time t_f. This trajectory passes through the same points as the original one, but with the time sense reversed. That is, if $\vec{R}(t)$, $\vec{V}_C(t)$ are the position and velocity at time t on the original trajectory, and $\vec{R}^*(t)$, $\vec{V}_C^*(t)$ those for the reverse one, then $\vec{R}^*(t) = \vec{R}(t_f - t)$ for $0 \leq t \leq t_f$, and thus $\vec{V}_C^*(t) = -\vec{V}_C(t_f - t)$.

Now define V^*_{CR} to be the radial component of $\vec{V}^*_C$ at $\vec{R}_T$, and $V^*_{C\theta}$ the horizontal component in the direction of $\vec{R}$. (Thus $V^*_{C\theta}$ and V^*_{CR} are both positive.) Also, let V_{CRT}, $V_{C\theta T}$ be the radial and horizontal components of $\vec{V}(t_f)$. Clearly $V_{C\theta T} = V^*_{C\theta}$ and $V_{CRT} = -V^*_{CR}$. Since $V_{C\theta}\ R = \ell$ is constant,

$$V^*_{C\theta} = V_{C\theta T} = \delta V_{C\theta}\ , \text{ where } \delta = R/R_T\ . \tag{25}$$

Since this relationship continues to hold when t_f and θ_R are allowed to vary,

$$\frac{\partial V^*_{C\theta}}{\partial t_f} = \delta\ \frac{\partial V_{C\theta}}{\partial t_f}\ ;\ \frac{\partial V^*_{C\theta}}{\partial \theta_R} = \delta\ \frac{\partial V_{C\theta}}{\partial \theta_R}\ . \tag{26}$$

The relationship between $\partial V_{C\theta}/\partial R_T$ and $\partial V^*_{C\theta}/\partial R_T$ is not quite as simple since δ depends on R_T, but a simple application of the chain rule to Equation 25 yields

$$\frac{\partial V_{C\theta}}{\partial R_T} = \delta^{-1}\ \frac{\partial V^*_{C\theta}}{\partial R_T} + \frac{1}{R_T}\ V_{C\theta}\ . \tag{27}$$

Also, the same argument used to derive Equation 22 can be applied to the reverse trajectory to show that

$$V^*_{CR}\ \frac{\partial V^*_{C\theta}}{\partial R_T} = \frac{V^*_{C\theta}}{R_T}\left(\frac{\partial V^*_{C\theta}}{\partial \theta_R} - V^*_{CR}\right) + \frac{\partial V^*_{C\theta}}{\partial t_f}\ . \tag{28}$$

Substituting Equations 25, 26, and 27 into Equation 28 yields

$$\frac{\partial V_{C\theta}}{\partial R_T} = \frac{1}{R_T}\left(\delta\ \frac{V_{C\theta}}{V_{CR}{}^*}\ \frac{\partial V_{C\theta}}{\partial \theta_R} + \frac{R_T}{V_{CR}{}^*}\ \frac{\partial V_{C\theta}}{\partial t_f}\right)\ . \tag{29}$$

(For any practical case $V_{CRT} < 0$, so $V^*_{CR} = -V_{CRT}$ is never zero.)

The second relation involving $\partial V_{C\theta}/\partial R_T$ can be derived from dimensional considerations. Recall that $V_{C\theta} = g(t_f, R, R_T, \theta_R, K^2)$. The value of g does not depend on any physical constants other than the arguments themselves, so that this function can be regarded as a purely numerical relation among

the variables $V_{C\theta}$, t_f, θ_R, R, R_T, K^2. Thus, if the numerical values of t_f, R, R_T, and K^2 change because of a change in the physical units (i.e., units of distance and time), the change in the value of $V_{C\theta}$ will be the same as if the physical conditions had changed by the same amount. E.g., changing the time unit must have the same effect on the numerical value of $V_{C\theta}$ as the corresponding variations in the actual time of flight (t_f) and the strength of the gravity field (K^2).

Now suppose we choose fixed values t_{f_0}, θ_{R_0}, R_0, R_{T_0}, K^2_0, and $V_{C\theta_0} = g(t_{f0}, R_0, \ldots)$ and change the distance unit by a factor $1/\alpha$, so that all distances are multiplied by a factor α. Since K^2 has dimensions of $(\text{distance})^3/(\text{time})^2$, its value will be multiplied by α^3. g has dimensions of distance/time, so its value will be multiplied by α. Thus

$$\alpha V_{C\theta_0} = g\left(t_{f_0}, \alpha R_0, \alpha R_{T_0}, \theta_{R_0}, \alpha^3 K_0^2\right) . \tag{30}$$

If we now change the time unit by a factor $1/\beta$, similar reasoning yields

$$\frac{\alpha}{\beta} V_{C\theta_0} = g\left(\beta t_{f_0}, \alpha R_0, \alpha R_{T_0}, \theta_{R_0}, \frac{\alpha^3}{\beta^2} K_0^2\right) . \tag{31}$$

The value of K^2 must remain constant, since otherwise $\partial V_{C\theta}/\partial K^2$ will appear in the final result. This can be arranged by setting $\beta = \alpha^{3/2}$, giving

$$\alpha^{-1/2} V_{C\theta_0} = g\left(\alpha^{3/2} t_{f_0}, \alpha R_0, \alpha R_{T_0}, \theta_{R_0}, K_0^2\right) . \tag{32}$$

Differentiating with respect to α gives

$$\begin{aligned} -\frac{1}{2}\alpha^{-3/2} V_{C\theta_0} &= \frac{d}{d\alpha} g\left(\alpha^{3/2} t_{f_0}, \alpha R_0, \alpha R_{T_0}, \theta_{R_0}, K_0^2\right) \\ &= \frac{3}{2}\alpha^{1/2} t_{f_0} \frac{\partial g}{\partial t_f} + R_0 \frac{\partial g}{\partial R} + R_{T_0} \frac{\partial g}{\partial R_T} \quad , \end{aligned} \tag{33}$$

where the derivatives are evaluated at the point $(\alpha^{3/2} t_{f_0}, \alpha R_0, \alpha R_{T_0}, \theta_{R_0}, K_0^2)$.

An expression relating the derivatives at the point (t_{f_0}, R_0, R_{T0}, θ_{R0}, K_0^2) can be obtained by setting $\alpha = 1$:

$$-\frac{1}{2} V_{C\theta 0} = \frac{3}{2} t_{f0} \frac{\partial g}{\partial t_f} + R_0 \frac{\partial g}{\partial R} + R_{T0} \frac{\partial g}{\partial R_T} \quad . \tag{34}$$

Since the derivatives here are evaluated at the original point, the zero subscripts may be dropped and the convention of denoting $\partial g/\partial x$ by $\partial V_{C\theta}/\partial x$ ($x = t_f$, R, etc.) may be resumed. Thus Equation 34 can be rewritten

$$\frac{\partial V_{C\theta}}{\partial R_T} = -\frac{1}{R_T}\left(\frac{1}{2} V_{C\theta} + \frac{3}{2} t_f \frac{\partial V_{C\theta}}{\partial t_f} + R \frac{\partial V_{C\theta}}{\partial R}\right) \quad . \tag{35}$$

Eliminating $\partial V_{C\theta}/\partial R_T$ between Equations 29 and 35 yields

$$\delta \frac{V_{C\theta}}{V_{CR}{}^*} \frac{\partial V_{C\theta}}{\partial \theta_R} + \frac{1}{2} V_{C\theta} + \left(\frac{R_T}{V_{CR}{}^*} + \frac{3}{2} t_f\right)\frac{\partial V_{C\theta}}{\partial t_f} + R \frac{\partial V_{C\theta}}{\partial R} = 0 \quad . \tag{36}$$

Multiplying through by V_{CR} and replacing $V_{CR} \frac{\partial V_{C\theta}}{\partial R}$ by the right side of Equation 22 gives

$$V_{C\theta}\left(1 + \delta \frac{V_{CR}}{V_{CR}{}^*}\right) \frac{\partial V_{C\theta}}{\partial \theta_R} = \frac{1}{2} V_{C\theta} V_{CR} - \left[R + V_{CR}\left(\frac{R_T}{V_{CR}{}^*} + \frac{3}{2} t_f\right)\right] \frac{\partial V_{C\theta}}{\partial t_f} \quad . \tag{37}$$

It is shown in Appendix C that the coefficient of $\partial V_{C\theta}/\partial \theta_R$ in this equation is never zero for cases of interest. This equation can therefore be used to find $\partial V_{C\theta}/\partial \theta_R$ in terms of $\partial V_{C\theta}/\partial t_f$. Of course, V^*_{CR} must first be computed. The simplest way to do this seems to be to apply Equation 1 to the reverse trajectory. This gives

$$V^*_{CR} = \left[\left(\cos \theta_R - \frac{R_T}{R}\right) V^*_{C\theta} + \frac{K^2}{R_T V^*_{C\theta}} \left(1 - \cos \theta_R\right)\right] / \sin \theta_R \quad . \tag{38}$$

Applying Equation 25 gives the more convenient form

$$V^*_{CR} = \left[\left(\delta \cos \theta_R - 1\right) V_{C\theta} + \frac{K^2}{RV_{C\theta}}\left(1 - \cos \theta_R\right)\right] / \sin \theta_R \quad . \tag{39}$$

The derivation is now complete. The suggested algorithm for computing Q may be summarized as follows. $V_{C\theta}$ and $\partial V_{C\theta}/\partial t_f$ should be available from the $\vec{V}_C$ computation. V_{CR} and V^*_{CR} are found using Equations 1 and 39 respectively. Next $\partial V_{C\theta}/\partial\theta_R$ is computed from Equation 37. The partials of f are found from Equations 11, 12, and 13, and $\partial V_{CR}/\partial\theta_R$ from Equation 14. Now $\partial V_{C\theta}/\partial R$ can be computed from Equation 24, and $\partial V_{CR}/\partial R$ from Equation 15. Equation 10 then gives Q in the local coordinate frame. Finally, the symmetry of Q can be exploited to save time in transforming it to the desired frame.

APPENDIX A

COMPUTATION OF $\vec{V}_C$, $V_{C\theta}$, V_{CR}, AND $\partial V_{C\theta}/\partial t_f$

COMPUTATION OF $\vec{V}_C$, $V_{C\theta}$, V_{CR}, and $\partial V_{C\theta}/\partial t_f$

First, an algorithm will be derived for computing the value of t_f corresponding to a given value of $V_{C\theta}$ for specified initial and target vectors. This derivation essentially follows that given originally by T. Alexander. Some additional terminology must be introduced here:

$e \equiv$ eccentricity of this trajectory ellipse

$\ell \equiv$ specific angular momentum

$\theta \equiv$ true anomaly

In this section, motion is assumed to be along the trajectory ellipse. As usual, V_θ and V_R denote the horizontal and vertical components of velocity, respectively. However, the symbols $V_{C\theta}$ and V_{CR} will be reserved for the initial values of these variables. Initial values of other quantities will be denoted by zero subscripts.

Finally, we define $\theta_D \equiv \theta - \theta_0$. θ will be taken to be positive in the direction of motion, so that $\theta_D > 0$ for $t > 0$.

The usual formula for R is:

$$R = \frac{\ell^2/K^2}{1 + e\cos\theta} \quad . \tag{A1}$$

Since $\ell = RV_\theta = R^2\dot{\theta}$, differentiating this equation gives immediately:

$$V_R = \frac{K^2}{\ell} e \sin\theta \quad . \tag{A2}$$

Next, Equation (A1) can be rewritten in the form:

$$K^2R[1 + (e\cos\theta_0)\cos\theta_D - (e\sin\theta_0)\sin\theta_D] = \ell^2 = R_0^2V_{C\theta}^2 \ . \tag{A3}$$

Solving Equation A1 for $e \cos\theta_0$ and Equation A2 for $e \sin\theta_0$ and substituting the results into Equation A3 yields:

$$R[K^2 + (R_0 V_{C\theta}^2 - K^2)\cos\theta_D - R_0 V_{C\theta} V_{CR} \sin\theta_D] = R_0^2 \, V_{C\theta}^2 \, . \tag{A4}$$

Now at the target $R = R_T$ and $\theta_D = \theta_R$. Substituting these values into Equation A4 and solving for $R_0 V_{C\theta} V_{CR}$ gives:

$$R_0 V_{C\theta} V_{CR} = \left[\left(\cos\theta_R - \frac{R_0}{R_T}\right) R_0 V_{C\theta}^2 + K^2 (1 - \cos\theta_R)\right] / \sin\theta_R \, . \tag{A5}$$

Substituting the expression on the right for $R_0 V_{C\theta} V_{CR}$ in Equation A4 and setting $\beta = R_0 V_{C\theta}^2$, we have:

$$R[K^2 R_T \sin\theta_R + R_T \sin\theta_R (\beta - K^2)\cos\theta_D - \{R_T(\beta - K^2)\cos\theta_R$$

$$- R_0\beta + R_T K^2\}\sin\theta_D] = \beta R_0 \, R_T \sin\theta_R \, . \tag{A6}$$

This equation, unlike Equation A4, does not involve V_{CR}, and thus gives R explicitly as a function of θ_D and the boundary conditions $\vec{R}_0$, $\vec{R}_T$, and $V_{C\theta}$.

Now $\ell = R_0 V_{C\theta} = R^2 (d\theta_D/dt)$, or $R_0 V_{C\theta} dt = R^2 d\theta_D$, so that:

$$t_f = \frac{1}{R_0 V_{C\theta}} \int_0^{\theta R} R^2(\theta_D) d\theta_D \, . \tag{A7}$$

It remains only to evaluate the integral in this equation and simplify the resulting expression for t_f as much as possible. For this purpose, define:

$$A = K^2 R_T \sin\theta_R$$

$$B = R_T(\beta - K^2)\sin\theta_R$$

$$C = \beta(R_0 - R_T \cos\theta_R) - K^2 R_T(1 - \cos\theta_R)$$

$$D = (R_0\beta)^{3/2}(R_T \sin\theta_R)^2 \quad .$$

It can be verified easily that:

$$\frac{t_f}{D} = \int_0^{\theta R} \frac{d\theta}{(A + B\cos\theta_D + C\sin\theta_D)^2}$$

$$= \frac{-B\sin\theta_R + C\cos\theta_R}{F(A + B\cos\theta_R + C\sin\theta_R)} - \frac{C}{F(A + B)}$$

$$+ \frac{2A}{F^{3/2}} \tan^{-1}\left[\frac{F^{1/2}\, H \tan\tfrac{1}{2}\theta_R}{F + C(C + H\tan\tfrac{1}{2}\theta_R)}\right] \quad , \qquad \text{(A8)}$$

where

$$F = A^2 - B^2 - C^2,$$

$$H = A - B.$$

An efficient algorithm for evaluating this expression for t_f is given below. Although a detailed derivation will not be given, the following key relations were used:

$$A + B\cos\theta_R + C\sin\theta_R = R_0\ \beta \sin\theta_R$$

$$H\tan\tfrac{1}{2}\theta_R = R_T(2K^2 - \beta)(1 - \cos\theta_R) \qquad \text{(A9)}$$

$$F + C(C + H\tan\tfrac{1}{2}\theta_R) = R_T(2K^2 - \beta)R_T\ \beta\sin^2\theta_R + C(1 - \cos\theta_R) \ .$$

It is sometimes useful to know for what values of $V_{C\theta}$ a solution exists. The acceptable values are those on the open interval ($V_{C\theta min}$, $V_{C\theta max}$). Equations for $V_{C\theta min}$ and $V_{C\theta max}$ are also given below.

The algorithm is as follows. The value of $V_{C\theta}$ used is denoted $V^*_{C\theta}$ and the corresponding value of t_f by t^*_f. Also, R_0 is denoted simply by R, and $C_1 \equiv \cos\theta_R$, $C_2 \equiv \sin\theta_R$, and $\mu_I \equiv 1/K^2$.

Compute:

$$S_I = 1/C_2$$

$$b_1 = R/R_T$$

$$d_2 = 1 + b_1(b_1 - 2c_1)$$

$$b_5 = 1 + c_1$$

$$d_1 = \mu_I b_5 R$$

$$b_9 = 1 + b_1$$

$$\left\{V_{C\theta min}, V_{C\theta max} = c_2/\left[d_1\left(b_9 \pm \sqrt{b_9^2 - d_2}\right)\right]^{1/2}\right\}$$

$$b_6 = b_1 b_5$$

$$b_8 = \mu_I b_6 R_T^2$$

(*) $b_2 = S_I V^*_{C\theta}$

$$d_3 = d_1 b_2$$

$$b_7 = d_3 b_2$$

$$b_{10} = b_9 - b_7 d_2$$

$$b_{11} = b_7 b_9 - 1$$

$$b_{12} = b_{11} + b_7 b_{10}$$

If $b_{12} \leq 0$, $V^*_{C\theta} \not\in (V_{C\theta min}, V_{C\theta max})$, so there is no solution for this value of $V_{C\theta}$. Otherwise:

$$b_{13} = \sqrt{b_{12}}$$

$$d_4 = b_{11}/b_{13}$$

$d_5 = \pi/2 - \arctan(d_4)$ (This is the standard two-quadrant arctangent.)

$$b_{14} = b_6 b_7 d_5$$

$$d_6 = b_{12} b_{13}$$

$$d_7 = b_{10} b_{13} + 2b_{14}$$

$$t_f^* = b_2 b_8 d_7/d_6$$

This completes the algorithm for evaluating t_f from Equation A8. This equation can be differentiated with respect to $V_{C\theta}$ to obtain an expression for $\partial t_f/\partial V_{C\theta}$, and thus of $\partial V_{C\theta}/\partial t_f = (\partial t_f/\partial V_{C\theta})^{-1}$. Again without going through the algebraic details, an algorithm for evaluating this expression is given below. The resulting value of $(\partial V_{C\theta}/\partial t_f)$ is denoted $(\partial V_{C\theta}/\partial t_f)$.*

Compute:

$$c_7 = 2d_3 S_I$$

$$c_{12} = 2c_7 b_{10}$$

$$c_{13} = c_{12}/(2b_{13})$$

$$c_{14} = b_6 \left\{ c_7 d_5 + b_7 d_4 (d_4 c_{13} - c_7 b_9) / \left[b_{11} (1 + d_4^2) \right] \right\}$$

$$(\partial V_{C\theta}/\partial t_f)^* = d_6 / \left\{ b_8 \left[S_I d_7 + b_2 (b_{10} c_{13} + 2c_{14} - b_{13} c_7 d_2) \right] - t_f^* (c_{12} b_{13} + b_{12} c_{13}) \right\}$$

In general, the values given initially are $\vec{R}$, $\vec{R}_T$, K^2, and t_f, while the values required are $\vec{V}_C$ and possibly $V_{C\theta}$, V_{CR}, and $\partial V_{C\theta}/\partial t_f$. The algorithms above give t_f and $\partial V_{C\theta}/\partial t_f$ explicitly in terms of R, R_T, $\cos\theta_R$, $\sin\theta_R$, and $V_{C\theta}$. R, R_T, $\cos\theta_R$, and $\sin\theta_R$ can be found immediately from the given values, while $\vec{V}_C$ can be determined once $V_{C\theta}$ and V_{CR} are known. Also, V_{CR} can be computed for any given value of $V_{C\theta}$ from Equation 1 which can be written in the form:

$$V_{CR} = R_T/(b_8 b_2) + b_2 (c_1 - b_1) .$$

NOTE: Since the value of $V_{C\theta}$ finally accepted may not be the last value of $V_{C\theta}^*$, $b_2 = S_I V_{C\theta}$ must in general be recomputed before V_{CR} is evaluated.

Thus the problem is reduced to that of finding the value of $V_{C\theta}$ corresponding to the given value of t_f. The required value must be found by iteration. Since t_f is a well-behaved function of $V_{C\theta}$ and $\partial V_{C\theta}/\partial t_f$ is available, the Newton-Raphson method is an obvious possibility. However, since $A \equiv (\partial V_{C\theta}/\partial t_f)t_f^2/\sin\theta_R$ is much more nearly constant than $\partial V_{C\theta}/\partial t_f$, more rapid convergence can be obtained by choosing

$$V_{C\theta}^{*} + (\partial V_{C\theta}/\partial t_f)^{*}(t_f^{*}/t_f)(t_f - t_f^{*}) \ ,$$

where

t_f is the given value of time of flight,

as the next estimate of $V_{C\theta}$ on each iteration.

Only the formulas from (*) on need to be reevaluated on each iteration.

If no better initial estimate of $V_{C\theta}$ is available, the value

$$V_{C\theta} = V_{C\theta min} + R_T \sin\theta_R/t_f$$

is usually satisfactory.

Lastly, if the final value of $V_{C\theta}$ is not the last value of $V_{C\theta}^{*}$ used in the algorithm, a better estimate of the corresponding value of $\partial V_{C\theta}/\partial t_f$ than $(\partial V_{C\theta}/\partial t_f)^{*}$ is

$$\partial V_{C\theta}/\partial t_f = (\partial V_{C\theta}/\partial t_f)^{*}(t_f^{*}/t_f)^2 \ .$$

APPENDIX B

DERIVATION OF EQUATION FOR $\dot{\vec{V}}_g$

DERIVATION OF EQUATION FOR $\dot{\vec{V}}_g$

The importance of the Q-matrix stems in large part from its occurrence in the well-known equation for $\dot{\vec{V}}_g$:

$$\dot{\vec{V}}_g = -\vec{A}_T - Q\vec{V}_g \quad , \tag{B1}$$

where $\vec{A}_T$ is the sensed (or gravity-free) acceleration; i.e., the acceleration produced by all forces except gravity. This result is established by the following simple argument.

First, with the conventions explained in the Introduction, clearly

$$\dot{\vec{V}}_C = Q\vec{V} - \frac{\partial \vec{V}_C}{\partial t_f} \quad . \tag{B2}$$

On the other hand, $\dot{\vec{V}} = \vec{A}_T + \vec{G}$ by definition of $\vec{A}_T$. An equation relating $\vec{G}$ to $\vec{V}_C$ and its derivatives can be obtained most simply by considering the "target" trajectory; i.e., the free-fall trajectory for which $\vec{V} = \vec{V}_C$ initially. As pointed out in the Introduction, on this trajectory $\vec{V} = \vec{V}_C$ for all t, so that $\dot{\vec{V}}_C = \dot{\vec{V}} = \vec{G}$. Combining this with the expression for $\dot{\vec{V}}_C$ given by Equation B2 yields

$$\vec{G} = Q\vec{V}_C - \frac{\partial \vec{V}_C}{\partial t_f} \quad . \tag{B3}$$

Now recall that $\vec{V}_C = \vec{V}_C(t_f, \vec{R}, \vec{R}_T, K^2)$, so that neither $\vec{V}_C$ nor any of its derivatives depend on the values of $\vec{V}$ or $\vec{A}_T$. Thus, while Equation B3 is most easily derived by assuming that $\vec{V} = \vec{V}_C$ and $\vec{A}_T = 0$, actually both sides are independent of $\vec{V}$ and $\vec{A}_T$, so that this equation is valid for any values of these variables. We can therefore write

$$\dot{\vec{V}} = \vec{A}_T + \vec{G} = \vec{A}_T + Q\vec{V}_C - \frac{\partial \vec{V}_C}{\partial t_f} \quad . \tag{B4}$$

Subtracting Equation B4 from Equation B2 yields Equation B1.

APPENDIX C

NECESSITY AND SUFFICIENCY OF EQUATIONS

NECESSITY AND SUFFICIENCY OF EQUATIONS

This section deals with the suitability of the algorithm developed in the main section and discusses alternative methods of computing the partials of $V_{C\theta}$.

The most important question to be considered is whether the coefficient of $\partial V_{C\theta}/\partial\theta_R$ in Equation 37 can ever be zero in any case of interest.[1] Since $V_{C\theta}$ is always positive, this is equivalent to asking whether it is possible that

$$1 + \frac{R}{R_T}\frac{V_{CR}}{V_{CR}^*} = 0 \tag{C1}$$

for such a case. Since $V_{CR}^* > 0$ for all such trajectories, clearly V_{CR} must be negative for Equation C1 to hold; i.e., the body must be past apogee.

Since $V_{CRT} = -V_{CR}^*$, Equation C1 is equivalent to

$$RV_{CR} = R_T V_{CRT} \quad . \tag{C2}$$

But by Equation 20, in any case

$$RV_{C\theta} = R_T V_{C\theta T} \quad . \tag{C3}$$

Squaring Equations C2 and C3 and adding gives

$$R^2 V_C^2 = R_T^2\, V_{CT}^2 \quad , \tag{C4}$$

where $V_{CT} = |\vec{V}_C(t_f)|$.

By the well-known vis-viva equation,

$$V_C^2 = K^2(2/R - 1/a) \quad , \tag{C5}$$

where a is the semimajor axis of the conic. Since a is constant and Equation C5 must hold at both $t = 0$ and $t = t_f$, this gives expressions for V_C and V_{CT} which can be substituted into Equation C4 to yield

$$(R - R_T)[2a - (R + R_T)] = 0 \quad . \tag{C6}$$

[1] The term "case of interest" refers here to conditions which might arise during powered flight for a ballistic missile.

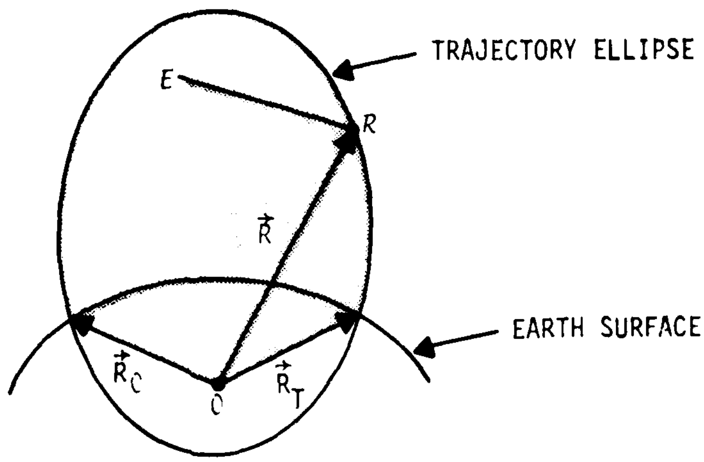
The only time R can equal R_T past apogee is at impact. At this point calculations are no longer required, so this case can be ignored. Otherwise the expression in brackets must be zero. Thus we must have $R + R_T = 2a$, or $a = \frac{1}{2}(R + R_T)$. We will show that computation of $\partial V_{C\theta}/\partial\theta_R$ will never be required past apogee for such a trajectory.

The sum of the distances from any point on an ellipse to the two foci is 2a. Thus the distance from $\vec{R}$ to the empty focus must be R_T. This is illustrated in Figure 1. But between apogee and impact $R > R_T$, so that $\vec{R}_T$ must lie on the same side of the minor axis as $\mathcal{O}$, the center of the earth. Now let $\vec{R}_0$ be the other point on the ellipse whose distance from $\mathcal{O}$ is R_T. The time required to go from $\vec{R}_0$ to $\vec{R}_T$ is, by Kepler's second law, $(A_S/A_E)P$ where A_S is the shaded area in the diagram, A_E is the total area of the ellipse, and P is the period. Clearly $A_S/A_E > 1/2$, and a well-known formula for P is

$$P = \frac{2\pi}{\sqrt{K^2}} a^{3/2} \tag{C7}$$

Now since $R > R_T$, $a = \frac{1}{2}(R + R_T) > R_T$, and since R_T must be essentially on the earth's surface we have $a > 2 \times 10^7$ ft. Also, $K^2 < 1.5 \times 10^{16}$ ft^3/sec^2, so $P > 4000$ sec. Thus the time required to go from $\vec{R}_0$ to $\vec{R}_T$ is at least 2000 sec, and the time to reach apogee is half that, or at least 1000 sec. Since the actual launch point is also on the earth, the time required to reach apogee on the actual trajectory is clearly greater than that required to reach apogee from $\vec{R}_0$ on the ellipse. But this formula will never be needed as late as 1000 seconds after launch. Since Equation C1 can only be satisfied past apogee, this shows that it cannot be satisfied in any case of interest.

This shows that Equation 37 can be used to compute $\partial V_{C\theta}/\partial\theta_R$ in a general algorithm. However, the question of whether Equation 24 can be used to produce a (possibly simpler) equation for this purpose should also be considered.



O = CENTER OF EARTH
E = EMPTY FOCUS OF TRAJECTORY ELLIPSE

$$\overline{OR}+\overline{RE} = 2a = R+R_T$$
$$\overline{OR} = R$$
$$\therefore \overline{RE} = R_T$$
$$\therefore \overline{RE} < \overline{OR}$$

Figure 1. Case Which Satisfies Equation C1

If $\partial V_{C\theta}/\partial R$ is eliminated between Equations 22 and 24, the result is

$$V_{C\theta} + V_{CR} \frac{\partial f}{\partial V_{C\theta}} \frac{\partial V_{C\theta}}{\partial \theta_R} = - V_{CR} \frac{\partial f}{\partial \theta_R} + R \frac{\partial V_{C\theta}}{\partial t_f} \tag{C8}$$

Thus the question is whether the coefficient of $\partial V_{C\theta}/\partial \theta_R$ in this equation can ever be zero; i.e., whether the equation

$$V_{C\theta} + V_{CR} \frac{\partial f}{\partial V_{C\theta}} = 0 \tag{C9}$$

can be satisfied in any case of interest. Since $\vec{V}_C = V_{CR} \vec{U}_1 + V_{C\theta} \vec{U}_2$, clearly

$$V_C^2 = V_{C\theta}^2 + V_{CR}^2 \quad . \tag{C10}$$

As explained in the DERIVATION section, Equation 2 defines the function $f(V_{C\theta}, R, R_T, \theta_R, K^2)$. Thus, by Equation C10, V_C can also be regarded as a function of these parameters. Differentiating with respect to $V_{C\theta}$ yields

$$V_C \frac{\partial V_C}{\partial V_{C\theta}} = V_{C\theta} + V_{CR} \frac{\partial f}{\partial V_{C\theta}} \quad . \tag{C11}$$

Thus, Equation C9 is satisfied if and only if $\partial V_C/\partial V_{C\theta} = 0$. This will happen for any value of $V_{C\theta}$ at which V_C attains a minimum (or maximum) value; in particular, for the value of $V_{C\theta}$ corresponding to the minimum energy trajectory. This is certainly a case of interest; in fact, a trajectory close to this one is often chosen deliberately to minimize the energy needed to reach the target.

(Clearly the minimum energy trajectory must be an ellipse. Each value of $V_{C\theta}$ on the open interval $(V_{C\theta_{min}}, V_{C\theta_{max}})$ corresponds to a unique elliptical path from $\vec{R}$ to $\vec{R}_T$, and vice-versa, so the fact that V_C is a minimum does in fact show that $\partial V_C/\partial V_{C\theta} = 0$.)

We have shown that Equation C8 in itself is not suitable for use in a general-purpose algorithm for computing Q. Still, it might be possible to use Equation C8 together with Equation 37 to determine $\partial V_{C\theta}/\partial \theta_R$ and $\partial V_{C\theta}/\partial t_f$ simultaneously. This would certainly be useful in some circumstances. For

example, in some applications a good estimate of $V_{C\theta}$ may be available without using the usual $\vec{V}_C$ iteration; in this case $\partial V_{C\theta}/\partial t_f$ is not automatically available.

Equations 28 and C8 can be combined into the matrix equation

$$\begin{bmatrix} V_{C\theta}\left(1 + \delta \dfrac{V_{CR}}{V_{CR}^*}\right) & \left[R + V_{CR}\left(\dfrac{R_T}{V_{CR}^*} + \dfrac{3}{2}\right)t_f\right] \\ V_{C\theta} + V_{CR}\dfrac{\partial f}{\partial V_{C\theta}} & R \end{bmatrix} \begin{bmatrix} \dfrac{\partial V_{C\theta}}{\partial \theta_R} \\ \dfrac{\partial V_{C\theta}}{\partial t_f} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} V_{C\theta} V_{CR} \\ -V_{CR}\dfrac{\partial f}{\partial \theta_R} \end{bmatrix} \tag{C12}$$

This system can be solved if the determinant of the matrix on the left is not zero. Thus the question to be considered is whether the equation

$$RV_{C\theta}\left(1 + \delta \frac{V_{CR}}{V_{CR}^*}\right) = \left(V_{C\theta} + V_{CR}\frac{\partial f}{\partial V_{C\theta}}\right)\left[R + V_{CR}\left(\frac{R_T}{V_{CR}^*} + \frac{3}{2} t_f\right)\right] \tag{C13}$$

can be satisfied for any cases of interest. This equation can be satisfied, but it is not known whether any situation in which it holds could actually occur in practice.

A further question is whether still another relationship among $\partial V_{C\theta}/\partial R$, $\partial V_{C\theta}/\partial \theta_R$, and $\partial V_{C\theta}/\partial t_f$ independent of those derived here might not be found. Several fairly obvious ways to derive such relations besides those described in this paper can be applied, but they all lead to relations which follow from those already obtained. On reflection, it can be seen that this was to be expected. Methods along the lines of those given here will lead to <u>linear</u> relations among these derivatives. Another such relation would make it possible to eliminate <u>all</u> the derivatives of $V_{C\theta}$ to give an algebraic expression relating t_f, $V_{C\theta}$, V_{CR}, R, R_T, K^2, sin θ_R, and cos θ_R. The occurrences of V_{CR} could then be eliminated by applying Equation 1. The result would be an algebraic relation between t_f and $V_{C\theta}$ in terms of the other parameters listed.

But this impossible, because it would lead to the result, for example, that d_4 and arctan d_4 are also related in this way. (See Appendix A.)

Nevertheless, there are some problems to be resolved. One is to find a useful characterization of the conditions under which Equation C13 is satisfied. Another is to determine whether a relation between $\partial V_{C\theta}/\partial \theta_R$ and $\partial V_{C\theta}/\partial t_f$ can be found which does not involve t_f explicitly, and in which the coefficient of $\partial V_{C\theta}/\partial \theta_R$ is never zero. (This would be useful in certain applicatons). Similarly, alternative expressions relating the derivatives of $V_{C\theta}$, though equivalent to those used here, might be of such a nature as to permit simultaneous solution for all cases of interest (or ideally, for all cases). These questions are appropriate subjects for further research.